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Solutions of the nonlinear wave equation $u_{tt} = (uu_x)_x$ invariant under conditional symmetries

M V Foursov and E M Vorob'ev†

Department of Applied Mathematics, Moscow Institute of Electronics and Mathematics, 3/12 B. Vouzovskii per., Moscow 109028, Russia

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Abstract. In this paper we discuss the application of conditional symmetries for obtaining exact and approximate solutions of the equation $u_{tt} = (uu_x)_x$. We propose to use the invariance properties of the infinitesimal conditional symmetries with respect to the classical Lie symmetries in order to explicitly solve the determining equations for the conditional symmetries. Exact solutions invariant under the conditional symmetries are given by explicit formulae, approximate solutions are calculated and presented in the graphical form by using 'Mathematica'.

1. Introduction

To date, significant progress in application of symmetries to analysis of nonlinear differential equations can be observed. The classical Lie symmetries of PDEs allow us to find explicit solutions, conservation laws, linearizing substitutions of the Hopf–Cole type, etc [1–5]. Unfortunately, for a lot of important applications differential equations the classical Lie symmetry groups are rather trivial including at most space and time translations and scale transformations. This stimulates the efforts devoted to generalization of Lie's original concept of symmetry [11]. Nonclassical conditional symmetries considered in this article are the classical symmetries of the new system of PDEs obtained by adding differential constraints, sometimes called 'side-conditions', the original system of PDEs. Conditional symmetries are associated with to Bäcklund transformations, functionally invariant solutions, and the 'direct' methods for explicit solution of PDEs [6].

The first approach to these symmetries was made by Bluman and Cole [7]. It consists in augmenting the original PDE with invariant surface conditions, a system of first order differential equations satisfied by all functions invariant under a certain vector field. The latter is chosen as a classical infinitesimal symmetry for the augmented system. The basic equations for conditional symmetries are similar to Lie's determining equations except that their number is less. That is why one rarely succeeds in obtaining all possible solutions to the determining equations for conditional symmetries, especially in the case of second order equations. As a rule, only particular solutions can be found, although many efforts are devoted to solving this problem by using differential Gröbner bases [8] and triangularization algorithms [9].

We must mention here an interesting problem of interplaying the classical and conditional symmetries. In fact, the determining equations for the conditional symmetries

† Supported in part by the grant 95-01-01079 of the Russian Foundation for Fundamental Research. E-mail address: emv@amath.msk.ru

inherit the classical symmetries of the original PDEs [10] allowing one to use the classical symmetries for systematic solution of nonclassical determining equations. In this paper, we use this approach for the nonlinear wave equation $u_{tt} = (uu_x)_x$.

Obtaining the classical and nonclassical symmetries of differential equations follows the algorithm involving prolongation of vector fields, restriction of functions and vector fields to submanifolds, reduction of overdetermined systems to passive forms and so on. Unfortunately, all these steps require simple but sometimes enormously large calculations. It is not surprising that designing symmetry symbolic packages has drawn considerable attention [13, 14]. We use a 'Mathematica' program called SYMMAN [15]. It provides a wide spectrum of symbolic symmetry calculations such as obtaining determining equations for classical and nonclassical symmetries and graphical representation of invariant solutions.

2. Physical phenomena described by the nonlinear wave equation $u_{tt} = (uu_x)_x$

The equation considered belongs to the family of nonlinear wave equations

$$u_{tt} = (k(u)u_x)_x \tag{1}$$

used by several authors for analysing nonlinear phenomena in continuous media. Zabusky [16] proposed the equation

$$v_{tt} = k(v_x)v_{xx} \tag{2}$$

for describing the dynamics of a nonlinear string. equation (2) can be transformed to (1) by differentiation w.r.t. the variable x followed by a substitution $v_x = u$.

Consider the equations of one-dimensional gas dynamics

$$\rho_t + (\rho v)_x = 0 \qquad v_t + vv_x + \rho^{-1}p_x = 0 \qquad p = p(\rho). \tag{3}$$

If we introdice the stream function $\psi(t, x)$ by the relations

$$\rho = \psi_x \qquad -\rho v = \psi_t$$

and take the variables t and ψ as our new independent coordinates, we obtain equation (1) for the function $u = \rho^{-1}$:

$$u_{tt} = \frac{\partial}{\partial\psi}\left(-\frac{\mathrm{d}p}{\mathrm{d}u}u_\psi\right).$$

Longitudinal wave propagation on a moving threadline under the assumption that the transverse vibrations are small is described by the system of equations

$$V_t + VV_x = \frac{\lambda}{\rho_0}\sigma_x \qquad \lambda_t + V\lambda_x - \lambda V_x = 0 \qquad \sigma = E_0 r(\lambda) \tag{4}$$

proposed in [17]. System (4) can be reduced to the equation

$$\lambda_{tt} = \mu^2\left(r'(\lambda)\lambda_\psi\right)_\psi \tag{5}$$

with $\mu^2 = E_0/\rho_0$. Finally, the equations of an electromagnetic transmission line

$$i_t + L^{-1}v_x = 0 \qquad v_t + C^{-1}i_x = 0 \qquad C = C(v)$$

can be transformed to the equation

$$v_{xx} = (LC(v)v_t)_t \tag{6}$$

for the voltage function $v(t, x)$. In the cases when the function $r(\lambda)$ is quadratic or the function $C(v)$ is linear, equations (5) and (6) take the form

$$u_{tt} = (uu_x)_x. \tag{7}$$

This equation is the subject of the present article.

3. Determining equations for the infinitesimal conditional symmetries

Equation (7) admits four-dimensional Lie algebra $\mathfrak{g}$ of its infinitesimal Lie symmetries [17] with a basis:

$$X_1 = \partial_t \qquad X_2 = \partial_x \qquad X_3 = t\partial_t + x\partial_x \qquad X_4 = t\partial_t - 2u\partial_u. \tag{8}$$

The corresponding one-parameter groups are time and space translations and scale transformations.

To discuss the conditional symmetries, we will treat equation (7) from a geometric point of view as a hypersurface E in the space J^2 of 2-jets of local functions $f(t, x)$ defined on the space R^2 of independent variables t, x [2]. Let

$$\boldsymbol{v} = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u \tag{9}$$

be a vector field on the space $R^2 \times R^1$ with coordinates t, x, and u. Then all functions invariant under $\boldsymbol{v}$ and only such functions satisfy a first order differential equation

$$\tau u_t + \xi u_x = \phi \tag{10}$$

called the invariant surface condition $E_{\boldsymbol{v}}$.

Denote by $E_{\boldsymbol{v}}^{(1)} \subset J^2$ the first prolongation of $E_{\boldsymbol{v}}$. The system $E_{\boldsymbol{v}}^{(1)}$ consists of (10) and the equations obtained by t- and x-differentiation of (10). Denote also by $\boldsymbol{v}^{(2)}$ the second prolongation of $\boldsymbol{v}$ to the space J^2.

Definition. A vector field $\boldsymbol{v}$ such that $\tau^2 + \xi^2 \not\equiv 0$ is called an infinitesimal conditional symmetry of equation (7) if $\boldsymbol{v}^{(2)}$ is tangent to the intersection $E \cap E_{\boldsymbol{v}}^{(1)}$.

The infinitesimal criterion of the tangency takes the form

$$\boldsymbol{v}^{(2)}(u_{tt} - (uu_x)_x)|_{E\cap E_{\boldsymbol{v}}^{(1)}} = 0. \tag{11}$$

Relation (11) is a generalization of the tangency condition for the classical Lie symmetries. It yields the determining equations for the coefficients τ, ξ, ϕ of the vector field $\boldsymbol{v}$.

It is well known that a vector field of conditional symmetries can be normalized by dividing it by any of its nonvanishing coefficients. So vector fields of the conditional symmetries can be split into two classes:

(1) vector fields with $\tau \not\equiv 0$ and, consequently, with $\tau = 1$,

(2) vector fields with $\tau \equiv 0$, $\xi \not\equiv 0$ and, consequently, with $\tau = 0$, $\xi = 1$.

We begin by considering the vector fields of the first class:

$$\boldsymbol{v} = \partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u. \tag{12}$$

For vector fields (12) it is possible to eliminate the derivatives u_t, u_{tt}, u_{tx}, and u_{xx} from (11) by solving the system $E \cap E_{\boldsymbol{v}}^{(1)}$ with respect to these derivatives and by substituting the expressions obtained into (11). After elimination, equation (11) becomes a third-order polynomial in the derivative u_x. Setting the coefficients of the polynomial equal to zero yields the determining equations for the functions ξ and ϕ. The program SYMMAN calculates the coefficients of the prolongation $\boldsymbol{v}^{(2)}$ of the vector field (12) onto the manifold J^2 supplied with the coordinates $t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}$ following the well known formulae [1–3]. Then it calculates the application of $\boldsymbol{v}^{(2)}$ to the function $u_{tt} - u_x^2 - uu_{xx}$, solves the system $E \cap E_{\boldsymbol{v}}^{(1)}$ w.r.t. the derivatives u_t, u_{tt}, u_{tx}, and u_{xx}, substitutes them into the expression $\boldsymbol{v}^{(2)}(u_{tt} - (uu_x)_x)$, takes the coefficients of the powers of u_x in the obtained expression and equates these coefficients to zero, which results in the following system of four determining equations:

$$
\begin{aligned}
&\phi^2\phi_u - 2\phi^2\xi\phi_u\xi_u - u\phi^2\phi_{uu} + \phi^2\xi^2\phi_{uu} - \phi\xi\phi_x + 2u\phi\xi_u\phi_x - 2u\phi\phi_u\xi_x + 2u\xi\phi_x\xi_x \\
&\qquad + u^2\phi_{xx} - u\xi^2\phi_{xx} + \phi\phi_t - 2\phi\xi\xi_u\phi_t - 2u\xi_x\phi_t - 2\phi\xi\phi_u\xi_t + 2u\phi_x\xi_t \\
&\qquad - 2\xi\phi_t\xi_t - 2u\phi\phi_t u + 2\phi\xi^2\phi_{tu} - u\phi_{tt} + \xi^2\phi_{tt} = 0 \\
&-2\phi\xi\phi_u - \phi^2\xi_u + 4\phi\xi^2\phi_u\xi_u + 2\phi^2\xi\xi_u^2 + 2u\phi\xi\phi_{uu} - 2\phi\xi^3\phi_{uu} + u\phi^2\xi_{uu} - \phi^2\xi^2\xi_{uu} \\
&\qquad + 2u\phi_x - 2\xi^2\phi_x + \phi\xi\xi_x + 4u\xi\phi_u\xi_x - 2u\xi\xi_x^2 + 2u^2\phi_{xu} - 2u\xi^2\phi_{xu} \\
&\qquad - u^2\xi_{xx} + u\xi^2\xi_{xx} - 2u\xi_u\phi_t + 2\xi^2\xi_u\phi_t - \phi\xi_t + 2u\phi_u\xi_t + 2\xi^2\phi_u\xi_t \\
&\qquad + 4\phi\xi\xi_u\xi_t + 2\xi\xi_t^2 + 2u\xi\phi_{tu} - 2\xi^3\phi_{tu} + 2u\phi\xi_{tu} - 2\phi\xi^2\xi_{tu} \\
&\qquad + u\xi_{tt} - \xi^2\xi_{tt} = 0 \\
&-\phi + u\phi_u - \xi^2\phi_u + 4\phi\xi\xi_u + 2u\xi\phi_u\xi_u - 2\xi^3\phi_u\xi_u - 4\phi\xi^2\xi_u^2 + u^2\phi_{uu} - 2u\xi^2\phi_{uu} \\
&\qquad + \xi^4\phi_{uu} - 2u\phi\xi\xi_{uu} + 2\phi\xi^3\xi_{uu} + 2\xi^2\xi_x - 4u\xi\xi_u\xi_x - 2u^2\xi_{xu} \\
&\qquad + 2u\xi^2\xi_{xu} + 2\xi\xi_t - 4\xi^2\xi_u\xi_t - 2u\xi\xi_{tu} + 2\xi^3\xi_{tu} = 0 \\
&u\xi_u - \xi^2\xi_u - 2u\xi\xi_u^2 + 2\xi^3\xi_u^2 - u^2\xi_{uu} + 2u\xi^2\xi_{uu} - \xi^4\xi_{uu} = 0.
\end{aligned}
\tag{13}
$$

For the vector fields of the second class

$$\boldsymbol{v} = \partial_x + \phi(t, x, u)\partial_u \tag{14}$$

the determining equations

$$\phi_{uu} = 0 \qquad \phi_{tu} = 0 \qquad \phi_{tt} - u\phi_{xx} - 2u\phi\phi_{xu} - 3\phi\phi_x - 2\phi^2\phi_u = 0$$

imply that

$$\phi(t, x, u) = a(x)u + b(t, x) \tag{15}$$

where the functions $a(x)$ and $b(t, x)$ satisfy the overdetermined system:

$$
\begin{aligned}
&b_{tt} - 3bb_x - 2ab^2 = 0 \qquad b_{xx} + 3ab_x + 5a'b + 4a^2b = 0 \\
&a'' + 5aa' + 2a^3 = 0.
\end{aligned}
\tag{16}
$$

The next two sections deal with solving systems (13) and (16).

4. Infinitesimal conditional symmetries of second type

The first two equations of (16) imply the compatibility condition

$$14ab_x^2 + 46a'bb_x + 36a^2bb_x - 37ab^2a' - 18a^3b^2 = 0 \tag{17}$$

obtained by cross-differentiation. The second equation of (16) is a differential consequence of (17) provided the equation

$$(60a' - 48a^2)b_x^2 - (510aba' + 348a^3b)b_x - 267b^2a'^2 - 233a^2b^2a' - 70a^4b^2 = 0 \tag{18}$$

is satisfied. After elimination of b_x from (17) and (18) we come to the equation

$$b^3(2a^2 + a')(302\,707a^6 + 1616\,526a^4a' + 2869\,917a^2a'^2) = 0. \tag{19}$$

If we take the third factor in (19) and the equation for the function $a(x)$ in (16), then that overdetermined system for the function $a(x)$ admits the unique solution $a(x) = 0$. Hence, in this case $b(t, x) = c_1(t)x + c_2(t)$, where the functions $c_1(t)$ and $c_2(t)$ satisfy the system of ODE:

$$\ddot{c}_1 - 3c_1^2 = 0 \qquad \ddot{c}_2 - 3c_1c_2 = 0. \tag{20}$$

Particular solutions of system (20) are $c_1(t) = 2/t^2$, $c_2(t) = c_{21}t^3 + c_{22}t^{-2}$. They yield an exact explicit solution of equation (1) obtainable by solving equation (10), which is an ordinary differential equation in the variable x for the symmetries of second type, and subsequent solution of equation (1) for the 'constants' of integration actually depending on the variable t:

$$u(t, x) = \frac{x^2}{t^2} + \frac{c_{22}x}{t^2} + c_{21}t^3x + \frac{c_{22}^2}{4t^2} + \frac{c_{21}c_{22}t^3}{2} + \frac{c_{21}^2t^8}{54} + \frac{c_{23}}{t} + c_{24}t^2 \tag{21}$$

where c_{2i} are parameters. In the generic case, the family of solutions (21) consists of solutions that are not invariant under any classical symmetry. But there are particular values of the parameters c_{ij} making the corresponding solutions invariant. Precisely, there are four such choices for c_{ij}:

solutions are invariant under the two-dimensional subalgebra generated by the vector fields X_4 and $c_{22}X_2 + 2X_3$ if $c_{21} = c_{23} = c_{24} = 0$,

solutions are invariant under $c_{22}X_2 + 2X_3 - X_4$ if $c_{21} = c_{23} = 0$,

solutions are invariant under $c_{22}X_2 + 2X_3 + 2X_4$ if $c_{21} = c_{24} = 0$,

solutions are invariant under $5c_{22}X_2 + 10X_3 - 8X_4$ if $c_{23} = c_{24} = 0$.

If we set $b(t, x) = 0$, then first two equations of (16) are satisfied and we arrive at an infinitesimal conditional symmetry

$$\boldsymbol{v} = \partial x + a(x)u\partial u$$

with the function $a(x)$ satisfying the ODE $a'' + 5aa' + 2a^3 = 0$. Particular solution $a(x) = 2/x$ of the latter equation yields exact solutions of the nonlinear wave equation

$$u(t, x) = w(t)x^2 \qquad \text{with } w(t) \text{ satisfying } \ddot{w} - 6w^2 = 0 \tag{22}$$

while particular solution $a(x) = 1/(2x)$ yields the exact solution

$$u(t, x) = c_1t\sqrt{x}. \tag{23}$$

Actually, solutions (22) and (23) are invariant under classical symmetry vector fields $X_3 - X_4$ and $2X_3 - X_4$, respectively.

Finally, if we take the second factor $a' + 2a^2 = 0$ in (19), which implies the third equation of (16), we arrive at an infinitesimal conditional symmetry

$$\boldsymbol{v} = \partial x + \left(\frac{u}{2x} + c(t)x\right)\partial u \tag{24}$$

where the function $c(t)$ satisfies the ODE $\ddot{c} - 4c^2 = 0$. If the take the particular solution $c(t) = 3/(2t^2)$, we obtain the exact solution

$$u(t, x) = \frac{x^2}{t^2} + \left(\frac{c_1}{t^{3/2}} + c_2t^{5/2}\right)\sqrt{x} \tag{25}$$

of equation (1). This solution is invariant under the classical infinitesimal symmetry $3X_3 - 2X_4$ if $c_1 = 0$, and it is invariant under $X_3 + 2X_2$ if $c_2 = 0$.

5. Infinitesimal conditional symmetries of first type

The last equation of system (13) contains only derivatives of the function $\xi(t,x,u)$ with respect to the variable u. It admits a classical infinitesimal scale symmetry $\boldsymbol{v}=2u\partial u+\xi\partial\xi$, so we can obtain the solution $\xi=\eta(t,x)\sqrt{u}$ to that equation. After substituting this expression for the function ξ into (13), we get that $\eta(t,x)=\pm 1$. We did not succeed in finding the general solution for the function $\phi(t,x,u)$ in the considered case. Particular solution $\phi(t,x,u)=c$ with c constant yields vector fields $\boldsymbol{v}=\partial t\pm\sqrt{u}\partial x+c\partial u$. The invariant surface condition takes the form

$$u_t\pm\sqrt{u}u_x=c. \tag{26}$$

In the case $c=0$ equations (26) are intermediate integrals of (1) meaning that each solution of (26) is a solution of (1). If we consider the following Cauchy problem for equation (1): $u|_{t=0}=\Phi(x)$, $u_t|_{t=0}=\Psi(x)$, the solutions of (26) correspond to the initial data satisfying the relations $\Psi(x)=\mp\sqrt{\Phi(x)}\Phi'(x)$. For $c\neq 0$ the simulteneous solutions for (1), (26) are $u(t,x)=ct+c_1$.

In what follows, we restrict ourselves to infinitesimal conditional symmetries

$$\boldsymbol{v}=\partial t+\xi(t,x)\partial x+(f(t,x)u+g(t,x))\partial u \tag{27}$$

with special dependence on the variable u admissible by equation (1). equations (13) imply the following relations:

$$f_x=0 \qquad \xi_{xx}=0 \qquad g=-f\xi^2+2\xi^2\xi_x+2\xi\xi_t.$$

Hence,

$$f(t,x)=a(t) \qquad \xi(t,x)=b(t)x+c(t) \tag{28}$$

and the problem of obtaining the infinitesimal conditional symmetries (27) reduces to solving a system of 15 nonlinear ordinary differential equations of the third order for the unknown functions $a(t)$, $b(t)$, and $c(t)$, which is a consequence of (13). Due to nonlinearity of this system, its involutive form splits into several subsystems. As a rule, the latter cannot be solved explicitly. Here is an example of such a subsystem:

$$a'+a^2+13ab+24b^2=0 \qquad b'-5b^2-2ab=0 \qquad c'=0.$$

The last equation $c'=0$ appears explicitly in all subsystems we obtained provided $b(t)\not\equiv 0$. It reflects the simple theoretic fact that if $\boldsymbol{v}$ is a conditional symmetry, then the vector fields obtained from it by x-translations are also conditional symmetries. So we can make a conjecture that $c(t)$ is proportional to $b(t)$ if $b(t)\not\equiv 0$ and, moreover, in this case the function $c(t)$ may be set equal to zero with no loss of generality. Therefore, we will consider the vector fields of conditional symmetries

$$\boldsymbol{v}=\partial t+b(t)x\partial x+(a(t)u+x^2(2b(t)^3+2b(t)b'-a(t)b^2)\partial u \tag{29}$$

or, in the case $b(t)\equiv 0$,

$$\boldsymbol{v}=\partial t+c(t)\partial x+(a(t)u+2c(t)c'(t)-a(t)c(t)^2)\partial u. \tag{30}$$

We could continue our analysis by trying to obtain particular solutions of the subsystems. Instead, we prefer a more systematic method of using the symmetry properties of infinitesimal conditional symmetries with respect to the classical Lie symmetries.

Suppose that X is an infinitesimal classical symmetry of equation (1). Denote by $\exp(\tau X)$ the one-parameter transformation group on the space R^3 of the points (t,x,u)

associated with X. This group generates the induced actions $\exp(\tau X)^*$ on the space $C^\infty(R^3)$ of smooth functions and $\exp(\tau X)_*$ on the space $D(R^3)$ of vector fields on R^3:

$$\begin{aligned}&\exp(\tau X)^*(F)(t,x,u) = F \circ \exp(\tau X)(t,x,u)\\ &\exp(\tau X)_*(\boldsymbol{v}) = \exp(-\tau X)^* \circ \boldsymbol{v} \circ \exp(\tau X)^*\end{aligned} \tag{31}$$

where $F(t,x,u)$ is a smooth function, $\boldsymbol{v}$ is a vector field on R^3 considered as a first order linear differential operator on $C^\infty(R^3)$. It was demonstrated in theorems 4 and 6 of [10] that if $\boldsymbol{v}$ is an infinitesimal conditional symmetry of a differential equation and X is an infinitesimal classical symmetry of the same equation, then $\exp(\alpha X)_*(\boldsymbol{v})$ is also an infinitesimal symmetry.

The infinitesimal conditional symmetries $\boldsymbol{v}$ invariant under the vector field X must satisfy the commutation relation

$$[X, \boldsymbol{v}] = \lambda(t,x,u)\boldsymbol{v}. \tag{32}$$

Similarly to invariant solutions of differential equations, the invariant vector fields of conditional symmetries break up into conjugacy classes. Vector fields belonging to the same class can be obtained from a particular vector field of the class by transformations (31). Therefore, it is sufficient to obtain the vector fields invariant under the representatives of the conjugacy classes of the Lie algebra $\mathfrak{g}$ of the classical symmetries with respect to inner automorphisms. The standard methods of obtaining conjugacy classes of subalgebras under inner automorphisms [1] lead to the following list of representatives of conjugacy classes (optimal subalgebras) of the Lie algebra $\mathfrak{g}$ with the basis (8):

$$\begin{aligned}&\mathfrak{g}_1 = L(X_1) \qquad \mathfrak{g}_2 = L(X_2) \qquad \mathfrak{g}_3 = L(X_1 - X_2) \qquad \mathfrak{g}_4 = L(X_1 + X_2)\\ &\mathfrak{g}_5 = L(X_3) \qquad \mathfrak{g}_6 = L(X_4) \qquad \mathfrak{g}_7 = L(X_4 + X_2) \qquad \mathfrak{g}_8 = L(X_4 - X_3)\\ &\mathfrak{g}_9 = L(X_4 - X_3 + X_1) \qquad \mathfrak{g}_{10} = L(X_4 - X_3 - X_1)\\ &\mathfrak{g}_{11} = L(\zeta X_3 + X_4) \qquad \zeta \neq -1\end{aligned} \tag{33}$$

where $L(X_i)$ denotes the linear span of X_i. It is evident that for the infinitesimal conditional symmetries (29) invariant under $X_1 = \partial_t$ the functions $a(t)$ and $b(t)$ are constant while the coefficients of a vector field (27) invariant under $X_2 = \partial x$ do not depend on x. Let us find infinitesimal symmetries (29) invariant under $X_3 = t\partial_t + x\partial_x$. After calculating the commutator of vector fields X_3 and $\boldsymbol{v}$, we obtain the relation

$$\begin{aligned}[X_3, \boldsymbol{v}] = \partial t - tb'(t)x\partial x + (x^2(2ab^2 - 4b^3 + tb^2a' -\\ 4bb' + 2tabb' - 6tb^2b' - 2tb'^2 - 2tbb'') - tua')\partial u.\end{aligned} \tag{34}$$

Comparing (29) and (34) we see that the functions $a(t)$ and $b(t)$ must satisfy the ODEs $-tb' = b$, $-ta' = a$, i.e. $a(t) = c_1/t$, $b(t) = c_2/t$ with c_1, c_2 constant. Therefore, the infinitesimal conditional symmetries invariant under X_3 must look like

$$\boldsymbol{v} = \partial t + \frac{c_2 x}{t}\partial x + \left(\frac{c_1 u}{t} + \frac{(2c_2^3 - 2c_2^2 - c_1 c_2^2)x^2}{t^3}\right)\partial u. \tag{35}$$

Substituting (35) into determining equations (13), we arrive at the system of algebraic equations for the parameters c_1 and c_2. The algebraic system admits five solutions given along with the other invariant conditional symmetries in table 1.

Not all of subalgebras (33) are present in table 1. Some of them do not admit the infinitesimal conditional symmetries at all, some admit only classical symmetries or the same symmetries that are included as invariant under other subalgebras. Vector field $\boldsymbol{v}_{2,2}$ was obtained in [18], $\boldsymbol{v}_{2,3}$ and $\boldsymbol{v}_{5,2}$ in [12].

Table 1. Invariant conditional symmetries.

$\mathfrak{g}_2$	$v_{2,1} = \partial t + c_1 t^2 \partial x + \left(\frac{u}{t} + 3c_1^2 t^3\right) \partial u$
	$v_{2,2} = \partial t + c_1 t \partial x + 2c_1^2 t \partial u$
	$v_{2,3} = \partial t + a(t) u \partial u$ with $a(t)$ satisfying the ODE
	$a'' + aa' - a^3 = 0$
$\mathfrak{g}_5, \mathfrak{g}_6$	$v_{5,1} = t^3 \partial t - x t^2 \partial x + (2t^2 u - 6x^2) \partial u$
	$v_{5,2} = t \partial t + u \partial u$
	$v_{5,3} = t^3 \partial t + x t^2 \partial x + (x^2 - t^2 u) \partial u$
	$v_{5,4} = 2t \partial t + (5 - \sqrt{13}) x \partial x + 2(3 - \sqrt{13}) u \partial u$
	$v_{5,5} = 2t \partial t + (5 + \sqrt{13}) x \partial x + 2(3 + \sqrt{13}) u \partial u$

6. Solutions invariant under the conditional symmetries of the first type

Solutions invariant under the classical and conditional symmetries satisfy the combined system (1) and (10). The traditional method of solving this system consists of constructing the functions invariant under the vector field v of infinitesimal symmetries by means of first integrals of v. Then such functions are substituted in the considered PDE yielding the quotient equation in fewer independent variables. There exists a direct and pure algebraic method for obtaining the quotient equations [19]. It is sufficient to consider the restriction of system (1), (10) to the curve $\gamma = \{(t, x) | t = \phi(\alpha), x = \psi(\alpha)\}$ in the space R^2 of independent variables t, x. Under the assumption of transversality of v to γ, the restriction can be transformed so that one of the equations is an ODE in the variable α. As an example, consider the vector field $v_{2,2}$ and the curve γ determined by the equation $t = 0$. From equation (10)

$$u_t + c_1 t u_x = 2c_1^2 t$$

and its differential consequences it is possible to express the derivative u_{tt} via the x-derivatives of $u(t, x)$:

$$u_{tt} = 2c_1^2 - c_1 u_x + c_1^2 t^2 u_{xx}.$$

Substitiuting this expression into (1) and setting $t = 0$, we obtain the following ODE for the restrictions of the solutions to the curve γ:

$$(u u_x)_x = 2c_1^2 - c_1 u_x. \tag{36}$$

Let $u = f(x)$ be a solution of (36). Then the corresponding solution of equation (1) is obtained with the help of a one-parameter transformation group

$$\exp(\tau v)(t, x, u) = (t + \tau, x + t\tau + \tau^2/2, u + 2t\tau + \tau^2)$$

associated with $v_{2,2}$. We only need to eliminate the parameter α from the equations

$$t - \tau = 0 \qquad u - 2t\tau + \tau^2 = f(x - t\tau + \tau^2/2)$$

arriving at the solution

$$u(t, x) = f(x - t^2/2) + t^2. \tag{37}$$

Equation (36) admits the explicit solution (for simplicity, we set $c_1 = 1$):

$$f(x) = -x + \frac{2^{2/3} x^2}{(x^3 + a^3 + a^{3/2}\sqrt{2x + a}\sqrt{4x^2 - 2xa + a^2})^{1/3}} + \frac{(4x^3 + a^3 + a^{3/2}\sqrt{2x + a}\sqrt{4x^2 - 2xa + a^2})^{1/3}}{2^{2/3}} \tag{38}$$

with a a parameter. Relations (37) and (38) give the exact explicit solution of equation (1) obtained in [18]. It is invariant under the classical symmetries only if $a = 0$. The corresponding vector field of the classical infinitesimal symmetry is $X_4 - 2X_3$. Unfortunately, in the cases when it is possible to obtain explicit solutions invariant under the vector fields of table 1, they coincide with the solutions given by formulae (21), (23) or (25).

7. Approximate numerical invariant solutions

The method of obtaining the quotient equations described above provides a good opportunity for finding approximate, numerical invariant solutions of PDEs in two independent variables. For this purpose, it is enough to obtain the numerical solutions of the quotient equation and numerical solutions for the trajectories of the vector field of classical or conditional symmetry. To begin with, consider the classical infinitesimal scale symmetry $X_3 = t\partial t + x\partial x$. If we take the curve $\gamma = \{(t,x)|t = 1\}$, the quotient equation written on this curve looks like

$$((x^2 - u)u')' = 0. \tag{39}$$

The particular solution $u(x) = x^2$ of (39) yields the explicit unbounded solution $u(t,x) = x^2/t^2$ found in [17]. At the same time, equation (39) admits bounded solutions as the numerical calculation shows.

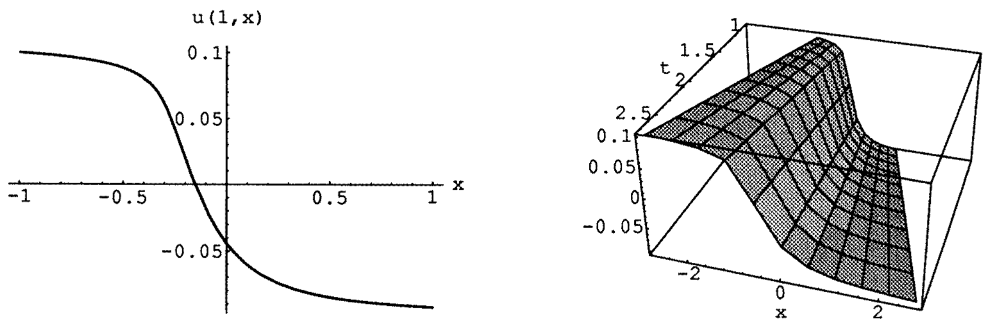

Figure 1. Bounded solution invariant under scale symmetry.

The left picture of figure 1 is a plot of the numerical solution of the Cauchy problem $u(-1) = 0.1$, $u'(-1) = -0.01$ for equation (39) on the interval $-1 \leqslant x \leqslant 1$ obtained by means of 'Mathematica' function NDSolve. The right picture is a plot of the corresponding invariant solution of the nonlinear wave equation (7) for $1 \leqslant t \leqslant 3$. We again used NDSolve for numerical integration of the equations for the trajectories of the vector field X_3 starting from the initial curve $\Gamma = \{(t,x,u)|t = 1, u = u(1,x)\}$.

The method of obtaining numerical solutions described above can be applied to solutions invariant under conditional symmetries. Consider the vector field $\boldsymbol{v}_{5,4}$ and the curve $\gamma = \{(t,x)|t = 1\}$. The quotient equation on γ for invariant solutions looks like:

$$(19x^2 - 5\sqrt{13}x^2 - 2u)u'' - 2u'^2 + (10\sqrt{13} - 32)xu' + (38 - 10\sqrt{13})u = 0. \tag{40}$$

Figure 2 shows the graphics of the solution of quotient equation (40) and the invariant solution corresponding to the Cauchy data $u(1,-2) = 1$, $u_x(1,-2) = 0$. We must note that the Cauchy data for equations (39) and (40) are set in such a way that the solutions lie far enough from the particular unbounded solutions $u = x^2$ of both equations. The plots of figures 1 and 2 are obtained with the help of functions QuotientEquations and InvarsolGraphics of program SYMMAN.

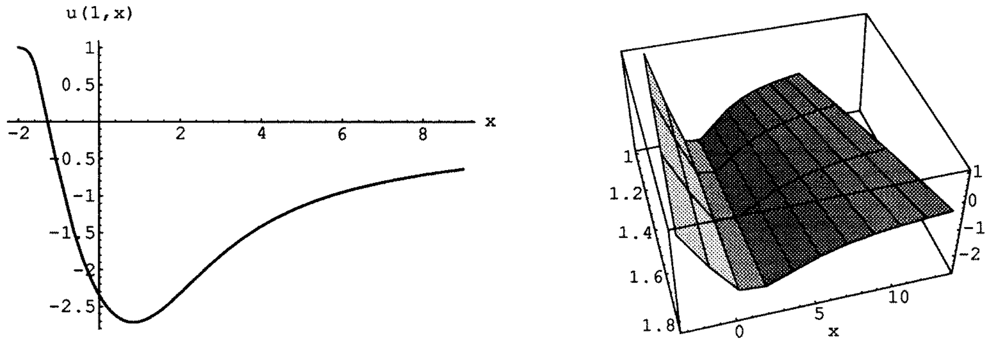

Figure 2. Bounded solution invariant under $v_{5,4}$.

8. Conclusions

The nonclassical conditional symmetries allow us to essentially increase the number of exact explicit solutions to the nonlinear wave equation $u_{tt} = (uu_x)_x$. Although the conditional symmetries of second class (14) yield more explicit solutions, the conditional symmetries of first class (12) admit invariant solutions with the properties of interest for applications. Therefore, the efforts to obtain particular solutions of the determining equations (13) were not useless. The invariance property of the conditional infinitesimal symmetries w.r.t. the classical Lie symmetries appears a useful tool for obtaining particular solutions of the determining equations. Using capabilities of 'Mathematica' for reliable symbolic computation, numerical solution of ODEs and graphical representation of approximate invariant solutions was very helpful though much must be done regarding the methods for solving the overdetermined systems of PDEs and the most informative plots of invariant solutions.

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